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ON A PAIR OF ZETA FUNCTIONS

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ABSTRACT. Let m be a positive integer, and define

$$\zeta_m(s) = \sum_{n=1}^{\infty} \frac{(-e^{2\pi i/m})^{\omega(n)}}{n^s} \quad \text{and} \quad \zeta_m^*(s) = \sum_{n=1}^{\infty} \frac{(-e^{2\pi i/m})^{\Omega(n)}}{n^s},$$

for $\Re(s) > 1$, where $\omega(n)$ denotes the number of distinct factors of n , and $\Omega(n)$ represents the total number of prime factors of n (counted with multiplicity). In this paper we study these two zeta functions and related arithmetical functions. We show that

$$\sum_{\substack{n=1 \\ n \text{ is squarefree}}}^{\infty} \frac{(-e^{2\pi i/m})^{\omega(n)}}{n} = 0 \quad \text{if } m > 4,$$

which is similar to the known identity $\sum_{n=1}^{\infty} \mu(n)/n = 0$ equivalent to the Prime Number Theorem. For $m > 4$, we prove that

$$\zeta_m(1) := \sum_{n=1}^{\infty} \frac{(-e^{2\pi i/m})^{\omega(n)}}{n} = 0 \quad \text{and} \quad \zeta_m^*(1) := \sum_{n=1}^{\infty} \frac{(-e^{2\pi i/m})^{\Omega(n)}}{n} = 0,$$

and that both $V_m(x)(\log x)^{2\pi i/m}$ and $V_m^*(x)(\log x)^{2\pi i/m}$ have explicit given finite limits as $x \rightarrow \infty$, where

$$V_m(x) = \sum_{n \leq x} \frac{(-e^{2\pi i/m})^{\omega(n)}}{n} \quad \text{and} \quad V_m^*(x) = \sum_{n \leq x} \frac{(-e^{2\pi i/m})^{\Omega(n)}}{n}.$$

We also raise a hypothesis on the parities of $\Omega(n) - n$ which implies the Riemann Hypothesis.

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1. INTRODUCTION

The Riemann zeta function $\zeta(s)$, defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } \Re(s) > 1,$$

plays a very important role in number theory. As Euler observed,

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \quad \text{for } \Re(s) > 1.$$

(In such a product we always let p run over all primes.) It is well-known that $\zeta(s)$ for $\Re(s) > 1$ can be continued analytically to a complex function which is holomorphic everywhere except for a simple pole at $s=1$ with residue 1. The famous Riemann Hypothesis asserts that if $0 \leq \Re(s) \leq 1$ and $\zeta(s) = 0$ then $\Re(s) = 1/2$. The Prime Number Theorem $\pi(x) \sim x/\log x$ (as $x \rightarrow \infty$) is actually equivalent to $\zeta(1+it) \neq 0$ for any nonzero real number t . (See, e.g., R. Crandall and C. Pomerance [CP, pp. 33-37].)

Let μ be the Möbius function. It is well known that

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1 \quad \text{for } \Re(s) > 1.$$

Also, either of $\sum_{n=1}^{\infty} \mu(n)/n = 0$ and $\sum_{n \leq x} \mu(n) = o(x)$ is equivalent to the Prime Number Theorem. (Cf. T. M. Apostol [A].)

The reader may consult [A] and [IR, pp. 18-21] for the basic knowledge of arithmetical functions and the theory of Dirichlet's convolution and Dirichlet series.

If $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ is squarefree, then $\mu(n) = (-1)^{\Omega(n)}$ depends on $\Omega(n)$ modulo 2, where $\Omega(n)$ denotes the number of all prime factors of n (counted with multiplicity). For the Liouville function $\lambda(n) = (-1)^{\Omega(n)}$, it is known that

$$\sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

Landau proved in his thesis that the equality $\sum_{n=1}^{\infty} \lambda(n)/n = 0$ is equivalent to the Prime Number Theorem. J. van de Lune and R. E. Dressler [LD] showed that $\sum_{n=1}^{\infty} (-1)^{\omega(n)}/n = 0$, where $\omega(n)$ denotes the number of distinct prime factors of n .

Now we give natural extensions of the functions $\mu(n)$, $\lambda(n)$ and $\zeta(s)$.

Definition 1.1. For $n \in \mathbb{Z}^+$ we set

$$\mu_m(n) = \begin{cases} (-e^{2\pi i/m})^{\omega(n)} & \text{if } n \text{ is squarefree,} \\ 0 & \text{otherwise,} \end{cases} \quad (1.1)$$

$$\nu_m(n) = (-e^{2\pi i/m})^{\omega(n)} \quad \text{and} \quad \nu_m^*(n) = (-e^{2\pi i/m})^{\Omega(n)}. \quad (1.2)$$

For $\Re(s) > 1$ we define

$$\zeta_m(s) = \sum_{n=1}^{\infty} \frac{\nu_m(n)}{n^s} = \prod_p \left(1 - \frac{e^{2\pi i/m}}{p^s - 1} \right) \quad (1.3)$$

and

$$\zeta_m^*(s) = \sum_{n=1}^{\infty} \frac{\nu_m^*(n)}{n^s} = \prod_p \left(1 + \frac{e^{2\pi i/m}}{p^s} \right)^{-1}. \quad (1.4)$$

As ν_m^* is completely multiplicative, the second identity in (1.4) is easy and in fact known. Since ν_m is multiplicative, if $\Re(s) > 1$ then

$$\sum_{n=1}^{\infty} \frac{\nu_m(n)}{n^s} = \prod_p \sum_{k=0}^{\infty} \frac{\nu_m(p^k)}{p^{ks}} = \prod_p \left(1 - e^{2\pi i/m} \sum_{k=1}^{\infty} \frac{1}{p^{ks}} \right)$$

and hence the second equality in (1.3) does hold.

As $\mu_1 = \mu$, we call μ_m the generalized Möbius function of order m . Note that $\zeta_2(s) = \zeta_2^*(s) = \zeta(s)$. Also, $\nu_1^*(n) = (-1)^{\Omega(n)}$ is the Liouville function $\lambda(n)$, and

$$\zeta_1^*(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)} = \prod_p \left(1 + \frac{1}{p^s} \right)^{-1} \quad \text{for } \Re(s) > 1.$$

If we replace $-e^{2\pi i/m}$ in the definition of $\zeta_m^*(s)$ by $e^{2\pi i/m}$, the resulted function was showed to have an infinitely many valued analytic continuation into the half plane $\Re(s) > 1/2$ by T. Kubota and M. Yoshida [KY]. (See also [A] and [CD].) It seems that the zeta function $\zeta_m(s)$ introduced here has not been studied before.

Our following theorem is not difficult.

Theorem 1.1. *Let m be any positive integer.*

(i) *The function $\mu_m^*(n) = \mu_m(n)\lambda(n)$ is the inverse of $\nu_m^*(n)$ with respect to the Dirichlet convolution, and hence*

$$\zeta_m^*(s) \sum_{n=1}^{\infty} \frac{\mu_m^*(n)}{n^s} = 1 \quad \text{for } \Re(s) > 1. \quad (1.5)$$

For $\Re(s) > 1$ we also have

$$\zeta_m(s) \sum_{n=1}^{\infty} \frac{(1 + e^{2\pi i/m})^{\Omega(n)}}{n^s} = \zeta(s). \quad (1.6)$$

(ii) If $m > 4$, then

$$\prod_p \left(1 + \frac{e^{2\pi i/m}}{p}\right)^{-1} = 0. \quad (1.7)$$

On the other hand,

$$\prod_p \left(1 + \frac{e^{2\pi i/3}}{p}\right) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} \left| \prod_{p \leq x} \left(1 + \frac{e^{2\pi i/4}}{p}\right) \right| = \frac{\sqrt{15}}{\pi}. \quad (1.8)$$

Remark 1.1. If $\Re(s) > 1$, then both $\zeta_m^*(s)$ and $\zeta_m(s)$ are nonzero by (1.5) and (1.6).

Our second theorem is a general result.

Theorem 1.2. *Let z be a complex number with $\Re(z) < 1$. Then*

$$\sum_{n \leq x} \frac{z^{\omega(n)}}{n} = \mathcal{F}(z)(\log x)^z + c(z) + O((\log x)^{z-1}) \quad (1.9)$$

and

$$\sum_{\substack{n \leq x \\ n \text{ is squarefree}}} \frac{z^{\omega(n)}}{n} = \mathcal{G}(z)(\log x)^z + c_*(z) + O((\log x)^{z-1}), \quad (1.10)$$

where $c(z)$ and $c_*(z)$ are constants only depending on z , and

$$\begin{aligned} \mathcal{F}(z) &= \frac{1}{\Gamma(1+z)} \prod_p \left(1 + \frac{z}{p-1}\right) \left(1 - \frac{1}{p}\right)^z, \\ \mathcal{G}(z) &= \frac{1}{\Gamma(1+z)} \prod_p \left(1 + \frac{z}{p}\right) \left(1 - \frac{1}{p}\right)^z. \end{aligned}$$

If $|z| < 2$, then

$$\sum_{n \leq x} \frac{z^{\Omega(n)}}{n} = \mathcal{H}(z)(\log x)^z + C(z) + O((\log x)^{z-1}), \quad (1.11)$$

where $C(z)$ is a constant only depending on z , and

$$\mathcal{H}(z) = \frac{1}{\Gamma(1+z)} \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z.$$

Theorem 1.2 obviously has the following consequence.

Corollary 1.1. *For any complex number z with $\Re(z) < 0$, we have*

$$\sum_{n=1}^{\infty} \frac{z^{\omega(n)}}{n} = c(z) \quad \text{and} \quad \sum_{\substack{n=1 \\ n \text{ is squarefree}}}^{\infty} \frac{z^{\omega(n)}}{n} = c_*(z). \quad (1.12)$$

If $|z| < 2$ and $\Re(z) < 0$, then

$$\sum_{n=1}^{\infty} \frac{z^{\Omega(n)}}{n} = C(z). \quad (1.13)$$

Theorem 1.3. *We have*

$$\sum_{n=1}^{\infty} \frac{\mu_5(n)}{n} = \sum_{n=1}^{\infty} \frac{\mu_6(n)}{n} = \dots = 0. \quad (1.14)$$

Moreover, for any positive integer $m \neq 2$ we have

$$(\log x)^{e^{2\pi i/m}} \sum_{n \leq x} \frac{\mu_m(n)}{n} = \mathcal{G}(-e^{2\pi i/m}) + O\left(\frac{1}{\log x}\right) \quad (x \geq 2), \quad (1.15)$$

where $\mathcal{G}(z)$ is defined as in Theorem 1.2.

Remark 1.2. It is known that

$$\sum_{n \leq x} \frac{\mu_2(n)}{n} = \sum_{n \leq x} \frac{|\mu(n)|}{n} = \frac{6}{\pi^2} \log x + c + O\left(\frac{1}{\sqrt{x}}\right) \quad (x \geq 2)$$

where $c = 1.04389 \dots$ (see, e.g., [BS, Lemma 14]). (1.15) with $m = 4$ implies that

$$\lim_{x \rightarrow \infty} \left| \sum_{n \leq x} \frac{\mu_4(n)}{n} \right| = |\mathcal{G}(-i)|.$$

After reading the first version of this paper, D. Broadhurst simplified $|\mathcal{G}(-i)|$ as $\sqrt{15 \sinh \pi / \pi^3}$.

Theorem 1.4. *Let*

$$V_m(x) = \sum_{n \leq x} \frac{\nu_m(n)}{n} \quad \text{and} \quad V_m^*(x) = \sum_{n \leq x} \frac{\nu_m^*(n)}{n}$$

for $m \in \mathbb{Z}^+$ and $x \geq 2$. Then

$$\begin{aligned} V_3(x) &= \mathcal{F}(-e^{2\pi i/3})(\log x)^{(1-i\sqrt{3})/2} + c_3 + O\left(\frac{1}{\sqrt{\log x}}\right), \\ V_3^*(x) &= \mathcal{H}(-e^{2\pi i/3})(\log x)^{(1-i\sqrt{3})/2} + C_3 + O\left(\frac{1}{\sqrt{\log x}}\right), \end{aligned} \quad (1.16)$$

and

$$\begin{aligned} V_4(x) &= \mathcal{F}(-i)(\log x)^{-i} + c_4 + O\left(\frac{1}{\log x}\right), \\ V_4^*(x) &= \mathcal{H}(-i)(\log x)^{-i} + C_4 + O\left(\frac{1}{\log x}\right), \end{aligned} \quad (1.17)$$

where c_3, C_3, c_4, C_4 are suitable constants. Also, for $m = 5, 6, \dots$ we have $V_m(x) = V_m^*(x) = o(1)$, i.e.,

$$\zeta_m(1) := \sum_{n=1}^{\infty} \frac{\nu_m(n)}{n} = 0 \quad \text{and} \quad \zeta_m^*(1) := \sum_{n=1}^{\infty} \frac{\nu_m^*(n)}{n} = 0. \quad (1.18)$$

Moreover, for $m = 1, 5, 6, \dots$ we have

$$V_m(x)(\log x)^{e^{2\pi i/m}} = \mathcal{F}(-e^{2\pi i/m}) + O\left(\frac{1}{\log x}\right) \quad (1.19)$$

and

$$V_m^*(x)(\log x)^{e^{2\pi i/m}} = \mathcal{H}(-e^{2\pi i/m}) + O\left(\frac{1}{\log x}\right). \quad (1.20)$$

Remark 1.3. It seems that c_3 and C_3 are nonzero but $c_4 = 0$ (and probably also $C_4 = 0$). Broadhurst simplified $|\mathcal{H}(-i)|$ as $\sqrt{(\sinh \pi)\pi/15}$.

Theorem 1.1 is not difficult. Our proofs of Theorems 1.2-1.4 depend heavily on some results of A. Selberg [S] (see also H. Delange [D] and Theorem 7.18 of [MV, p. 231]) and the Abel summation method via Abel's identity (see, [A, p. 77]).

Motivated by Theorem 1.4 we raise the following conjecture for further research.

Conjecture 1.1. Both $V_1(x) = \sum_{n \leq x} (-1)^{\omega(n)}/n$ and $V_1^*(x) = \sum_{n \leq x} (-1)^{\Omega(n)}/n$ are $O(x^{\varepsilon-1/2})$ for any $\varepsilon > 0$. Also, $|\sum_{n \leq x} (-2)^{\Omega(n)}| < x$ for all $x \geq 3078$.

Remark 1.4. It seems that $V_1(x)$ might be $O(\sqrt{(\log x)/x})$ or even $O(1/\sqrt{x})$. The asymptotic behavior of $\sum_{n \leq x} 2^{\Omega(n)}$ was investigated by E. Grosswald [G].

In 1958 C. B. Haselgrove [H] disproved Pólya's conjecture that $\sum_{n \leq x} \lambda(n) \leq 0$ for all $x \geq 2$; he also showed that Turán's conjecture $\sum_{n \leq x} \lambda(n)/n > 0$ for $x \geq 1$, is also false. (See also [L] and [BFM].) Our following hypothesis might be the right one along this direction.

Hypothesis 1.1. (i) *For any $x \geq 5$, we have*

$$S(x) := \sum_{n \leq x} (-1)^{n-\Omega(n)} > 0, \quad (1.21)$$

i.e.,

$$|\{n \leq x : \Omega(n) \equiv n \pmod{2}\}| > |\{n \leq x : \Omega(n) \not\equiv n \pmod{2}\}|.$$

Moreover,

$$S(x) > \sqrt{x} \text{ for all } x \geq 325, \text{ and } S(x) < 2.3\sqrt{x} \text{ for all } x \geq 1.$$

(ii) *For any $x \geq 1$ we have*

$$T(x) := \sum_{n \leq x} \frac{(-1)^{n-\Omega(n)}}{n} < 0. \quad (1.22)$$

Moreover,

$$T(x)\sqrt{x} < -1 \text{ for all } x \geq 2, \text{ and } T(x)\sqrt{x} > -2.3 \text{ for all } x \geq 3.$$

Remark 1.5. We have verified parts (i) and (ii) of the hypothesis for x up to 6×10^{10} and 2×10^9 respectively. Here are values of $S(x)$ for some particular x :

$$\begin{aligned} S(10) &= 2, \quad S(10^2) = 14, \quad S(10^3) = 54, \quad S(10^4) = 186, \quad S(10^5) = 464, \\ S(10^6) &= 1302, \quad S(10^7) = 5426, \quad S(10^8) = 19100, \quad S(10^9) = 62824, \\ S(10^{10}) &= 172250, \quad S(2 \cdot 10^{10}) = 252292, \quad S(3 \cdot 10^{10}) = 292154, \\ S(4 \cdot 10^{10}) &= 263326, \quad S(5 \cdot 10^{10}) = 360470, \quad S(5.5 \cdot 10^{10}) = 455216. \end{aligned}$$

Example 1.1. For $x_1 = 17593752$ and $x_2 = 123579784$, we have

$$S(x_1) = 9574, \quad S(x_2) = 11630, \quad \frac{S(x_1)}{\sqrt{x_1}} \approx 2.28252, \quad \frac{S(x_2)}{\sqrt{x_2}} \approx 1.04618.$$

Though we are unable to prove Hypothesis 1.1, we can show the following result.

Theorem 1.5. (i) *We have*

$$S(x) = o(x) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-\Omega(n)}}{n} = 0. \quad (1.23)$$

(ii) *If $S(x) > 0$ for all $x \geq 5$, or $T(x) < 0$ for all $x \geq 1$, then the Riemann Hypothesis holds.*

Note that

$$S(x) > 0 \iff |\{n \leq x : 2 \mid (n - \Omega(n))\}| > \frac{x}{2}.$$

In view of Hypothesis 1.1, it is natural to ask whether

$$|\{n \leq x : m \mid (n - \Omega(n))\}| > \frac{x}{m} \quad \text{for sufficiently large } x.$$

For $m = 3, 4, \dots, 18, 20$ we have the following conjecture.

Conjecture 1.2. *We have*

$$|\{n \leq x : 4 \mid (n - \Omega(n))\}| < \frac{x}{4} \quad \text{for any } x \geq s(4),$$

and for $m = 3, 5, 6, \dots, 18, 20$ we have

$$|\{n \leq x : m \mid (n - \Omega(n))\}| > \frac{x}{m} \quad \text{for all } x \geq s(m),$$

where

$$\begin{aligned} s(3) &= 62, \quad s(4) = 1793193, \quad s(5) = 187, \quad s(6) = 14, \quad s(7) = 6044, \quad s(8) = 73, \\ s(9) &= 65, \quad s(10) = 61, \quad s(11) = 4040389, \quad s(12) = 14, \quad s(13) = 6943303, \\ s(14) &= 4174, \quad s(15) = 77, \quad s(16) = 99, \quad s(17) = 50147927, \quad s(18) = 73, \quad s(20) = 61. \end{aligned}$$

Remark 1.7. The case $m = 19$ seems much more sophisticated. Perhaps the sign of $|\{n \leq x : 19 \mid (n - \Omega(n))\}| - x/19$ changes infinitely often.

As there are generalized Riemann Hypothesis for algebraic number fields, we propose the following extension of Hypothesis 1.1.

Hypothesis 1.2 (Generalized Hypothesis). *Let K be any algebraic number field. Then we have*

$$S_K(x) := \sum_{N(A) \leq x} (-1)^{N(A) - \Omega(A)} > 0 \quad \text{for sufficiently large } x,$$

where A runs over all nonzero integral ideals in K whose norm (with respect to the field extension K/\mathbb{Q}) are not greater than x , and $\Omega(A)$ denotes the total number of prime ideals in the factorization of A as a product of prime ideals (counted with multiplicity). In particular, for $K = \mathbb{Q}(i)$ we have $S_K(x) > 0$ for all $x \geq 9$, and for $K = \mathbb{Q}(\sqrt{-2})$ we have $S_K(x) > 0$ for all $x \geq 132$.

Now we give one more conjecture.

Conjecture 1.3. For an integer $d \equiv 0, 1 \pmod{4}$ define

$$S_d(x) = \sum_{n \leq x} (-1)^{n-\Omega(n)} \left(\frac{d}{n} \right),$$

where $\left(\frac{d}{n} \right)$ denotes the Kronecker symbol. Then

$$S_{-4}(x) < 0, \quad S_{-7}(x) < 0, \quad S_{-8}(x) < 0$$

for all $x \geq 1$, and

$$S_5(x) > 0 \text{ for } x \geq 11, \quad S_{-3}(x) > 0 \text{ for } x \geq 406759, \quad S_{-11}(x) > 0 \text{ for } x \geq 771862,$$

and

$$S_{24}(x) < 0 \text{ for } x \geq 90601, \quad \text{and} \quad S_{28}(x) < 0 \text{ for } x \geq 629819.$$

We will show Theorems 1.1 and 1.2 in the next section, and prove Theorems 1.3-1.5 in Sections 3-5 respectively.

2. PROOFS OF THEOREMS 1.1 AND 1.2

Proof of Theorem 1.1. Clearly $\mu_m^*(1)\nu_m^*(1) = 1 \cdot 1 = 1$. Let N be any integer greater than one, and let n be the product of all distinct prime factors of N . Then

$$\begin{aligned} \sum_{d|N} \mu_m^*(d) \nu_m^* \left(\frac{N}{d} \right) &= \sum_{d|n} e^{2\pi i \Omega(d)/m} (-e^{2\pi i/m})^{\Omega(n/d) + \Omega(N/n)} \\ &= (-1)^{\Omega(N/n)} e^{2\pi i \Omega(N)/m} \sum_{d|n} \mu \left(\frac{n}{d} \right) = 0. \end{aligned}$$

Therefore μ_m^* is the inverse of ν_m^* with respect to the Dirichlet convolution $*$.

Let $s = \sigma + it$ be a complex number with $\Re(s) = \sigma > 1$. Since

$$\max \left\{ \left| \frac{\mu_m^*(n)}{n^s} \right|, \left| \frac{\nu_m^*(n)}{n^s} \right| \right\} \leq \left| \frac{1}{n^{\sigma+it}} \right| = \left| \frac{e^{-it \log n}}{n^\sigma} \right| = \frac{1}{n^\sigma}$$

for any $n \in \mathbb{Z}^+$, both $\sum_{n=1}^{\infty} \mu_m^*(n)/n^s$ and $\sum_{n=1}^{\infty} \nu_m^*(n)/n^s$ converge absolutely. Therefore

$$\sum_{n=1}^{\infty} \frac{\mu_m^*(n)}{n^s} \sum_{n=1}^{\infty} \frac{\nu_m^*(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\mu_m^* * \nu_m^*(n)}{n^s} = 1.$$

Since $|p^s| = p^\sigma > p \geq |1 + e^{2\pi i/m}|$ for any prime p , we have

$$\prod_p \left(1 - \frac{1 + e^{2\pi i/m}}{p^s} \right)^{-1} = \prod_p \sum_{k=0}^{\infty} \frac{(1 + e^{2\pi i/m})^k}{p^{ks}} = \sum_{n=0}^{\infty} \frac{(1 + e^{2\pi i/m})^{\Omega(n)}}{n^s}.$$

Note that

$$\begin{aligned}\zeta_m(s) &= \prod_p \frac{p^s - 1 - e^{2\pi i/m}}{p^s - 1} = \prod_p \frac{1 - (1 + e^{2\pi i/m})/p^s}{1 - 1/p^s} \\ &= \zeta(s) \prod_p \left(1 - \frac{1 + e^{2\pi i/m}}{p^s}\right).\end{aligned}$$

So (1.6) does hold.

Now assume that $m > 4$. Then $2\pi/m < \pi/2$ and $0 < \cos(2\pi/m) < 1$. For any prime p we have

$$\left|1 + \frac{e^{2\pi i/m}}{p}\right| = \left|\left(1 + \frac{\cos(2\pi/m)}{p}\right) + i\frac{\sin(2\pi/m)}{p}\right| \geq 1 + \frac{\cos(2\pi/m)}{p}.$$

Therefore

$$\left|\prod_{p \leq x} \left(1 + \frac{e^{2\pi i/m}}{p}\right)\right| \geq \prod_{p \leq x} \left(1 + \frac{\cos(2\pi/m)}{p}\right) \geq 1 + \cos \frac{2\pi}{m} \sum_{p \leq x} \frac{1}{p},$$

and hence (1.7) holds since $\sum_p 1/p$ diverges (cf. [IR, p. 21]).

Finally we prove the first identity in (1.8). For any prime p , we have

$$\left|1 + \frac{e^{2\pi i/3}}{p}\right|^2 = 1 + 2\frac{\cos 2\pi/3}{p} + \frac{1}{p^2} = 1 - \frac{1}{p} + \frac{1}{p^2} = \frac{1 + p^{-3}}{1 + p^{-1}}.$$

Thus

$$\begin{aligned}\left|\prod_{p \leq x} \left(1 + \frac{e^{2\pi i/3}}{p}\right)\right|^2 &= \prod_{p \leq x} \left(1 + \frac{1}{p^3}\right) \cdot \prod_{p \leq x} \left(1 + \frac{1}{p}\right)^{-1} \\ &\leq \prod_p \left(1 + \frac{1}{p^3}\right) \cdot \left(1 + \sum_{p \leq x} \frac{1}{p}\right)^{-1}.\end{aligned}$$

Since $\sum_p 1/p$ diverges while $\sum_p 1/p^3$ converges, the first equality in (1.7) follows.

The second equality in (1.8) is easy. In fact, as $x \rightarrow \infty$,

$$\left|\prod_{p \leq x} \left(1 + \frac{e^{2\pi i/4}}{p}\right)\right|^2 = \prod_{p \leq x} \left|1 + \frac{i}{p}\right|^2$$

has the limit

$$\prod_p \left(1 + \frac{1}{p^2}\right) = \frac{\prod_p (1 - 1/p^2)^{-1}}{\prod_p (1 - 1/p^4)^{-1}} = \frac{\zeta(2)}{\zeta(4)} = \frac{\pi^2/6}{\pi^4/90} = \frac{15}{\pi^2}.$$

In view of the above, we have completed the proof of Theorem 1.1. \square

To prove Theorem 1.2, we need two lemmas.

Lemma 2.1 (Selberg [S]). *Let z be a complex number. For $x \geq 2$ we have*

$$\sum_{n \leq x} z^{\omega(n)} = F(z)x(\log x)^{z-1} + O\left(x(\log x)^{\Re(z)-2}\right) \quad (2.1)$$

and

$$\sum_{\substack{n \leq x \\ n \text{ is squarefree}}} z^{\omega(n)} = G(z)x(\log x)^{z-1} + O\left(x(\log x)^{\Re(z)-2}\right), \quad (2.2)$$

where

$$F(z) = \frac{1}{\Gamma(z)} \prod_p \left(1 + \frac{z}{p-1}\right) \left(1 - \frac{1}{p}\right)^z$$

and

$$G(z) = \frac{1}{\Gamma(z)} \prod_p \left(1 + \frac{z}{p}\right) \left(1 - \frac{1}{p}\right)^z.$$

When $|z| < 2$, for $x \geq 2$ we also have

$$\sum_{n \leq x} z^{\Omega(n)} = H(z)x(\log x)^{z-1} + O\left(x(\log x)^{\Re(z)-2}\right), \quad (2.3)$$

where

$$H(z) = \frac{1}{\Gamma(z)} \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z.$$

Lemma 2.2. *Let $a(1), a(2), \dots$ be a sequence of complex numbers. Suppose that*

$$\sum_{n \leq x} a(n) = cx(\log x)^{z-1} + O(x(\log x)^{\Re(z)-2}) \quad (x \geq 2), \quad (2.4)$$

where c and z are (absolute) complex numbers with $z \neq 0$ and $\Re(z) \neq 1$. Then, for $x, y \geq 2$ we have

$$\begin{aligned} \sum_{n \leq x} \frac{a(n)}{n} - \frac{c}{z}(\log x)^z - \left(\sum_{n \leq y} \frac{a(n)}{n} - \frac{c}{z}(\log y)^z \right) \\ = O((\log x)^{z-1}) + O((\log y)^{z-1}). \end{aligned} \quad (2.5)$$

Thus, if $\Re(z) < 1$ then

$$\sum_{n \leq x} \frac{a(n)}{n} = \frac{c}{z}(\log x)^z + c_z + O((\log x)^{\Re(z)-1}) \quad (x \geq 2), \quad (2.6)$$

where c_z is a suitable constant.

Proof. Let $A(t) = \sum_{n \leq t} a(n)$ for $t \geq 2$. By the Abel summation formula,

$$\begin{aligned} \sum_{n \leq x} \frac{a(n)}{n} - \sum_{n \leq y} \frac{a(n)}{n} &= \frac{A(x)}{x} - \frac{A(y)}{y} - \int_y^x A(t)(t^{-1})' dt \\ &= \frac{A(x)}{x} - \frac{A(y)}{y} + \int_y^x \frac{A(t)}{t^2} dt. \end{aligned}$$

Note that

$$\frac{A(t)}{t} = c(\log t)^{z-1} + O((\log t)^{\Re(z)-2}) \quad \text{for } t \geq 2.$$

Clearly

$$\int_y^x \frac{(\log t)^{z-1}}{t} dt = \left. \frac{(\log t)^z}{z} \right|_y^x = \frac{(\log x)^z - (\log y)^z}{z}$$

and

$$\int_y^x \frac{(\log t)^{\Re(z)-2}}{t} dt = \left. \frac{(\log t)^{\Re(z)-1}}{\Re(z)-1} \right|_y^x = \frac{(\log x)^{\Re(z)-1} - (\log y)^{\Re(z)-1}}{\Re(z)-1}.$$

So the desired (2.5) follows from the above.

Now assume that $\Re(z) < 1$. For any $\varepsilon > 0$ we can find a positive integer N such that for $x, y \geq N$ the absolute value of the right-hand side of (2.5) is smaller than ε . Therefore, in view of (2.5) and Cauchy's convergence criterion, $\sum_{n \leq x} a(n)/n - c(\log x)^z/z$ has a finite limit c_z as $x \rightarrow \infty$. Letting $y \rightarrow \infty$ in (2.5) we immediately obtain (2.6). This ends the proof. \square

Proof of Theorem 1.2. When $z = 0$, (1.9)-(1.11) obviously hold with $c(0) = c_*(0) = C(0) = 0$.

Now assume $z \neq 0$. As $\Gamma(1+z) = z\Gamma(z)$, we see that

$$\mathcal{F}(z) = \frac{F(z)}{z}, \quad \mathcal{G}(z) = \frac{G(z)}{z}, \quad \text{and} \quad \mathcal{H}(z) = \frac{H(z)}{z}.$$

Combining Lemmas 2.1 and 2.2 we immediately get the desired (1.9)-(1.11). \square

3. PROOF OF THEOREM 1.3

We first present two lemmas.

Lemma 3.1. *Let $m \in \mathbb{Z}^+$ and $x \geq 1$. Then we have*

$$\sum_{n \leq x} \mu_m(n) \left\lfloor \frac{x}{n} \right\rfloor = \sum_{n \leq x} (1 - e^{2\pi i/m})^{\omega(n)}. \quad (3.1)$$

Proof. We first claim that

$$\sum_{d|n} \mu_m(d) = (1 - e^{2\pi i/m})^{\omega(n)} \quad (3.2)$$

for any $n \in \mathbb{Z}^+$. Clearly (3.2) holds for $n = 1$. If $n = p_1^{a_1} \cdots p_k^{a_k}$ with p_1, \dots, p_k distinct primes and $a_1, \dots, a_k \in \mathbb{Z}^+$, then

$$\sum_{d|n} \mu_m(d) = \sum_{I \subseteq \{1, \dots, k\}} \mu_m \left(\prod_{i \in I} p_i \right) = \sum_{r=0}^k \binom{k}{r} (-e^{2\pi i/m})^r = (1 - e^{2\pi i/m})^{\omega(n)}.$$

Observe that

$$\sum_{d \leq x} \mu_m(d) \left\lfloor \frac{x}{d} \right\rfloor = \sum_{d \leq x} \mu_m(d) \sum_{q \leq x/d} 1 = \sum_{dq \leq x} \mu_m(d) = \sum_{n \leq x} \sum_{d|n} \mu_m(d).$$

Combining this with (3.2) we immediately obtain (3.1). \square

Lemma 3.2. *Let $m \in \mathbb{Z}^+$, $m \neq 2$, and $x \geq 2$. Then we have*

$$\sum_{n \leq x} \mu_m(n) \left\{ \frac{x}{n} \right\} = o(x), \quad \sum_{n \leq x} \nu_m(n) \left\{ \frac{x}{n} \right\} = o(x), \quad \sum_{n \leq x} \nu_m^*(n) \left\{ \frac{x}{n} \right\} = o(x), \quad (3.3)$$

where $\{\alpha\}$ denotes the fractional part of a real number α .

Proof. By (2.1)-(2.3),

$$\begin{aligned} \sum_{n \leq x} \mu_m(n) &= xG(-e^{2\pi i/m})(\log x)^{-e^{2\pi i/m}-1} + O\left(x(\log x)^{-\cos(2\pi/m)-2}\right) = o(x), \\ \sum_{n \leq x} \nu_m(n) &= xF(-e^{2\pi i/m})(\log x)^{-e^{2\pi i/m}-1} + O\left(x(\log x)^{-\cos(2\pi/m)-2}\right) = o(x), \\ \sum_{n \leq x} \nu_m^*(n) &= xH(-e^{2\pi i/m})(\log x)^{-e^{2\pi i/m}-1} + O\left(x(\log x)^{-\cos(2\pi/m)-2}\right) = o(x). \end{aligned}$$

(Note that $F(-1) = G(-1) = H(-1) = 0$.)

Let w be any of the three functions μ_m, ν_m, ν_m^* . By the above $W(n) = \sum_{n \leq x} w(n) = o(x)$. We want to show that

$$\Delta(x) := \sum_{n \leq x} w(n) \left\{ \frac{x}{n} \right\} = o(x).$$

Clearly

$$r(u) := \sup_{t \geq u} \frac{|W(t)|}{t} \leq 1 \quad \text{for } u \geq 1.$$

Also $r(u) \rightarrow 0$ as $u \rightarrow \infty$.

Let $0 < \varepsilon < 1$. Then

$$\begin{aligned}
|\Delta(x)| &\leq \left| \sum_{n \leq \varepsilon x} w(n) \left\{ \frac{x}{n} \right\} \right| + \left| \sum_{\varepsilon x < n \leq x} w(n) \left\{ \frac{x}{n} \right\} \right| \\
&\leq \varepsilon x + \left| \sum_{\varepsilon x < n \leq x} (W(n) - W(n-1)) \left\{ \frac{x}{n} \right\} \right| \\
&\leq \varepsilon x + \left| \sum_{\varepsilon x < n < \lfloor x \rfloor} W(n) \left(\left\{ \frac{x}{n} \right\} - \left\{ \frac{x}{n+1} \right\} \right) \right| \\
&\quad + \left| W(\lfloor x \rfloor) \left\{ \frac{x}{\lfloor x \rfloor} \right\} - W(\lfloor \varepsilon x \rfloor) \left\{ \frac{x}{\lfloor \varepsilon x \rfloor + 1} \right\} \right|.
\end{aligned}$$

Note that

$$\left| W(\lfloor x \rfloor) \left\{ \frac{x}{\lfloor x \rfloor} \right\} \right| \leq |W(\lfloor x \rfloor)| \frac{\{x\}}{\lfloor x \rfloor} \leq 1$$

and

$$\left| W(\lfloor \varepsilon x \rfloor) \left\{ \frac{x}{\lfloor \varepsilon x \rfloor + 1} \right\} \right| \leq |W(\lfloor \varepsilon x \rfloor)| \leq \lfloor \varepsilon x \rfloor \leq \varepsilon x.$$

Therefore

$$\begin{aligned}
|\Delta(x)| &\leq 1 + 2\varepsilon x + \sum_{\varepsilon x < n < \lfloor x \rfloor} \frac{|W(n)|}{n} x \left| \left\{ \frac{x}{n} \right\} - \left\{ \frac{x}{n+1} \right\} \right| \\
&\leq 1 + 2\varepsilon x + xr(\varepsilon x) \sum_{\varepsilon x < n < \lfloor x \rfloor} \left| \frac{x}{n} - \frac{x}{n+1} - \left(\left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n+1} \right\rfloor \right) \right| \\
&\leq 1 + 2\varepsilon x + xr(\varepsilon x) \sum_{\varepsilon x < n < \lfloor x \rfloor} \left(\left(\frac{x}{n} - \frac{x}{n+1} \right) + \left(\left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n+1} \right\rfloor \right) \right) \\
&\leq 1 + 2\varepsilon x + xr(\varepsilon x) \left(2 \frac{x}{\lfloor \varepsilon x \rfloor + 1} - \frac{x}{\lfloor x \rfloor} - \left\lfloor \frac{x}{\lfloor x \rfloor} \right\rfloor \right)
\end{aligned}$$

and hence

$$\frac{|\Delta(x)|}{x} \leq \frac{1}{x} + 2\varepsilon + \frac{2}{\varepsilon} r(\varepsilon x).$$

It follows that

$$\limsup_{x \rightarrow \infty} \frac{|\Delta(x)|}{x} \leq 2\varepsilon. \quad (3.4)$$

As (3.4) holds for any given $\varepsilon \in (0, 1)$, we must have $\Delta(x) = o(x)$ as desired. \square

Proof of Theorem 1.3. For $z = -e^{2\pi i/m}$ we have $\Re(z) = -\cos(2\pi/m) < 1$ since $m \neq 2$. Combining (2.1) with (2.3), we obtain

$$\sum_{n \leq x} \mu_m(n) \left\lfloor \frac{x}{n} \right\rfloor = F(1+z)x(\log x)^z + O\left(x(\log x)^{-1-\cos(2\pi/m)}\right).$$

By Lemma 3.2,

$$\sum_{n \leq x} \mu_m(x) \left\{ \frac{x}{n} \right\} = o(x).$$

Therefore

$$x \sum_{n \leq x} \frac{\mu_m(n)}{n} = \sum_{n \leq x} \mu_m(n) \left(\left\lfloor \frac{x}{n} \right\rfloor + \left\{ \frac{x}{n} \right\} \right) = F(1+z)x(\log x)^z + o(x)$$

and hence

$$\sum_{n \leq x} \frac{\mu_m(n)}{n} = \mathcal{G}(z)(\log x)^z + o(1) \quad (3.5)$$

since $F(1+z) = G(z)/z = \mathcal{G}(z)$. Combining (3.5) with (1.10) and noting that $(\log x)^{-z-1} \rightarrow 0$ as $x \rightarrow \infty$, we get $c_*(z) = 0$. So (1.10) reduces to (1.15).

For $m = 5, 6, \dots$ we clearly have $\cos(2\pi/m) > 0$ and hence (1.15) implies that $\sum_{n=1}^{\infty} \mu_m(n)/n = 0$. This concludes the proof. \square

Remark 3.1. The way we prove (1.14) can also be used to show Landau's equality $\sum_{n=1}^{\infty} \lambda(n)/n = 0$. Since $\lambda = \nu_1^*$, we have $\sum_{n \leq x} \lambda(n)\{x/n\} = o(x)$ by Lemma 3.2. So it suffices to prove $\sum_{n \leq x} \lambda(n)\lfloor x/n \rfloor = o(x)$. In fact,

$$\begin{aligned} \sum_{d \leq x} \lambda(d) \left\lfloor \frac{x}{d} \right\rfloor &= \sum_{d \leq x} \lambda(d) \sum_{q \leq x/d} 1 = \sum_{dq \leq x} \lambda(d) = \sum_{n \leq x} \sum_{d|n} \lambda(d) \\ &= |\{1 \leq n \leq x : n \text{ is a square}\}| = \lfloor \sqrt{x} \rfloor = o(x). \end{aligned}$$

4. PROOF OF THEOREM 1.4

Let $m \in \{1, 3, 4, \dots\}$ and $z = -e^{2\pi i/m}$. When $m = 3$, (1.9) and (1.11) yield (1.16) with $c_3 = c(z)$ and $C_3 = C(z)$. In the case $m = 4$, (1.9) and (1.11) give (1.17) with $c_4 = c(-i)$ and $C_4 = C(-i)$.

Below we assume that $m = 1$ or $m > 4$. Note that $\Re(z) = -\cos(2\pi/m) < 0$. By (1.9) and (1.11), we have

$$V_m(x) = \mathcal{F}(z)(\log x)^z + c_m + O((\log x)^{z-1})$$

and

$$V_m^*(x) = \mathcal{H}(z)(\log x)^z + C_m + O((\log x)^{z-1}),$$

where $c_m = c(z)$ and $C_m = C(z)$. If $c_m = C_m = 0$, then (1.19) and (1.20) follow. So it suffices to show $V_m(x) = o(1)$ and $V_m^*(x) = o(1)$. Note that $\zeta_1^*(1) = \sum_{n=1}^{\infty} \lambda(n)/n = 0$ by Landau's result (cf. Remark 3.1) and also $\zeta_1(1) = \sum_{n=1}^{\infty} (-1)^{\omega(n)}/n = 0$ by [LD].

Define $a(n) = z^{\Omega(n)} = \nu_m^*(n)$ for $n \in \mathbb{Z}^+$ and $A(x) := \sum_{n \leq x} a(n)$ for $x \geq 1$. And set $f_s(t) = t^{-s}$ for $s \geq 1$ and $t \geq 1$. By the Abel summation formula, for $x \geq x_0 \geq 2$ we have

$$\sum_{x_0 < n \leq x} a(n) f_s(n) = A(x) f_s(x) - A(x_0) f_s(x_0) - \int_{x_0}^x A(t) f'_s(t) dt$$

and hence

$$\sum_{x_0 < n \leq x} \frac{\nu_m^*(n)}{n^s} = \frac{A(x)}{x^s} - \frac{A(x_0)}{x_0^s} + s \int_{x_0}^x \frac{A(t)}{t^{s+1}} dt. \quad (4.1)$$

By (2.3), there is a positive constant C such that

$$|A(t) - H(z)t(\log t)^{z-1}| \leq Ct(\log t)^{-2-\cos(2\pi/m)} \quad \text{for all } t \geq 2.$$

Therefore

$$\begin{aligned} \left| \frac{A(x)}{x^s} \right| &\leq \left| \frac{H(z)(\log x)^{z-1}}{x^{s-1}} \right| + C \frac{(\log x)^{-2-\cos(2\pi/m)}}{x^{s-1}} \\ &\leq \frac{|H(z)|}{(\log x)^{1+\cos(2\pi/m)}} + \frac{C}{(\log x)^{2+\cos(2\pi/m)}}, \\ \left| \frac{A(x_0)}{x_0^s} \right| &\leq \frac{|H(z)|}{(\log x_0)^{1+\cos(2\pi/m)}} + \frac{C}{(\log x_0)^{2+\cos(2\pi/m)}}, \end{aligned}$$

and

$$\begin{aligned} \left| \int_{x_0}^x \frac{A(t)}{t^{s+1}} dt \right| &\leq \int_{x_0}^x \left| \frac{H(z)t(\log t)^{z-1}}{t^{s+1}} \right| dt + \int_{x_0}^x \left| \frac{A(t) - H(z)t(\log t)^{z-1}}{t^{s+1}} \right| dt \\ &\leq |H(z)| \int_{x_0}^x \frac{(\log t)^{-1-\cos(2\pi/m)}}{t} dt + C \int_{x_0}^x \frac{(\log t)^{-2-\cos(2\pi/m)}}{t} dt \\ &= |H(z)| \left| \frac{(\log t)^{-\cos(2\pi/m)}}{-\cos(2\pi/m)} \right|_{t=x_0}^x + C \left| \frac{(\log t)^{-1-\cos(2\pi/m)}}{-1-\cos(2\pi/m)} \right|_{t=x_0}^x \\ &= \frac{|H(z)|}{\cos(2\pi/m)} \left(\frac{1}{(\log x_0)^{\cos(2\pi/m)}} - \frac{1}{(\log x)^{\cos(2\pi/m)}} \right) \\ &\quad + \frac{C}{1+\cos(2\pi/m)} \left(\frac{1}{(\log x_0)^{1+\cos(2\pi/m)}} - \frac{1}{(\log x)^{1+\cos(2\pi/m)}} \right). \end{aligned}$$

Let $\varepsilon > 0$. If x and x_0 are large enough then by the above for any $s \geq 1$ we have

$$\left| \sum_{x_0 < n \leq x} \frac{\nu_m^*(n)}{n^s} \right| = \left| \frac{A(x)}{x^s} \right| + \left| \frac{A(x_0)}{x_0^s} \right| + s \left| \int_{x_0}^x \frac{A(t)}{t^{s+1}} dt \right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + s\varepsilon = (1+s)\varepsilon.$$

Therefore the series $\sum_{n=1}^{\infty} \nu_m^*(n)/n^s$ converges for any $s \geq 1$, in particular $\sum_{n=1}^{\infty} \nu_m^*(n)/n$ converges! If N is large enough, then for any $s \geq 1$ we have

$$\left| \sum_{n>N} \frac{\nu_m^*(n)}{n^s} \right| < (1+s)\varepsilon \quad \text{and} \quad \left| \sum_{n>N} \frac{\nu_m^*(n)}{n} \right| < 2\varepsilon,$$

and hence

$$\begin{aligned} \left| \sum_{n=1}^{\infty} \frac{\nu_m^*(n)}{n^s} - \zeta_m(1) \right| &\leq \left| \sum_{n=1}^N \left(\frac{\nu_m^*(n)}{n^s} - \frac{\nu_m^*(n)}{n} \right) \right| \\ &\quad + \left| \sum_{n>N} \frac{\nu_m^*(n)}{n^s} \right| + \left| \sum_{n>N} \frac{\nu_m^*(n)}{n} \right| \\ &\leq \left| \sum_{n=1}^N \left(\frac{\nu_m^*(n)}{n^s} - \frac{\nu_m^*(n)}{n} \right) \right| + (3+s)\varepsilon. \end{aligned}$$

Letting $s \rightarrow 1+$, we get $|\zeta_m^*(s) - \zeta_m^*(1)| \leq 3\varepsilon$. Therefore

$$\lim_{s \rightarrow 1+} \zeta_m^*(s) = \zeta_m^*(1). \quad (4.2)$$

Similarly, we have $\lim_{s \rightarrow 1+} \zeta_m(s) = \zeta_m(1)$.

Note that Theorem 1 of [KY] remains true if we use $z = -e^{2\pi i/m}$ instead of $\rho = e^{2\pi i/m}$ in [KY]. Thus, there is a function $\psi(s)$ holomorphic in the half plane $\Re(s) > 1/2$ such that

$$\zeta_m^*(s) \zeta(s)^{-z} = \psi(s).$$

As $\lim_{s \rightarrow 1} \psi(s) = \psi(1)$ and $\lim_{s \rightarrow 1+} \zeta(s)^{-z} = \infty$ we must have $\lim_{s \rightarrow 1+} \zeta_m^*(s) = 0$ and hence $\zeta_m^*(1) = 0$. We can modify the proof of [KY, Theorem 1] slightly to prove a similar result for $\zeta_m(s)$ and hence deduce $\zeta_m(1) = 0$.

So far we have completed the proof of Theorem 1.4.

5. PROOF OF THEOREM 1.5

Proof of Theorem 1.5. Let $L(x) = \sum_{n \leq x} (-1)^{\Omega(n)}$. (2.3) with $z = -1$ yields that $L(x) = o(x)$. Observe that

$$S(x) + L(x) = \sum_{n \leq x} ((-1)^n + 1)(-1)^{\Omega(n)} = 2 \sum_{m \leq x/2} (-1)^{\Omega(2m)} = -2L\left(\frac{x}{2}\right).$$

Therefore

$$S(x) = -L(x) - 2L\left(\frac{x}{2}\right) = o(x).$$

Clearly

$$\sum_{n \leq x} \frac{(-1)^{n-\Omega(n)}}{n^s} + \sum_{n \leq x} \frac{\lambda(n)}{n^s} = 2 \sum_{\substack{n \leq x \\ 2|n}} \frac{\lambda(n)}{n^s} = -2 \sum_{m \leq x/2} \frac{\lambda(m)}{(2m)^s}$$

and hence

$$\sum_{n \leq x} \frac{(-1)^{n-\Omega(n)}}{n^s} = -2^{1-s} \sum_{n \leq x/2} \frac{\lambda(n)}{n^s} - \sum_{n \leq x} \frac{\lambda(n)}{n^s}.$$

Since $\sum_{n \leq x} \lambda(n)/n = o(1)$ as shown by Landau, we get $\sum_{n \leq x} (-1)^{n-\Omega(n)}/n = o(1)$ and hence $\sum_{n=1}^{\infty} (-1)^{n-\Omega(n)}/n = 0$.

Let $\Re(s) > 1$. Note that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-\Omega(n)}}{n^s} = -(1 + 2^{1-s}) \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = -(1 + 2^{1-s}) \frac{\zeta(2s)}{\zeta(s)}.$$

On the other hand, by Abel's summation method, we have

$$\sum_{n \leq x} \frac{(-1)^{n-\Omega(n)}}{n^s} = \frac{S(x)}{x^s} + s \int_1^x \frac{S(t)}{t^{s+1}} dt$$

and hence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-\Omega(n)}}{n^s} = s \int_1^{\infty} \frac{S(t)}{t^{s+1}} dt.$$

Therefore

$$-(1 + 2^{1-s}) \frac{\zeta(2s)}{\zeta(s)} = s \int_1^{\infty} \frac{S(t)}{t^{s+1}} dt. \quad (5.1)$$

Let σ_c be the least real number such that the integral in (5.1) converges whenever $\Re(s) > \sigma_c$. By the above, $\sigma_c \leq 1$.

Suppose that $S(x) > 0$ for all $x \geq 5$. In view of (5.1), by applying Landau's theorem (cf. [MV, Lemma 15.1] or Ex. 16 of [Ap, p.248]) we obtain

$$\lim_{s \rightarrow \sigma_c} -\frac{1 + 2^{1-s}}{s} \cdot \frac{\zeta(2s)}{\zeta(s)} = \infty$$

and hence $\sigma_c \leq 1/2$ since $\zeta(s)$ has no real zeroes with $s > 1/2$. So the right-hand side of (5.1) converges for $\Re(s) > 1/2$ and hence so is the left-hand side of (5.1). Therefore $\zeta(s) \neq 0$ for $\Re(s) > 1/2$, i.e., the Riemann Hypothesis holds.

Similarly, if $T(x) < 0$ for all $x \geq 1$, then we get the Riemann Hypothesis by applying Landau's theorem.

So far we have completed the proof of Theorem 1.5. \square

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